Calculus 140, section 3.6 Implicit Differentiation

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All of the equations encountered so far have been functions, $y = f(x)$: for example $y = 45x^2 - x^3$ and $P(x) = \frac{80}{2 + 3e^{-10x}}$. This is an *explicit* statement of the function formula, and given an explicit function and a value for x, the determination of the corresponding y-coordinate becomes a calculation. In addition, the determination of the slope of the curve at that value of x means using one of the derivative rules developed so far, then calculating.

Even when a function is not expressed explicitly, it is sometimes possible to solve for the explicit version:

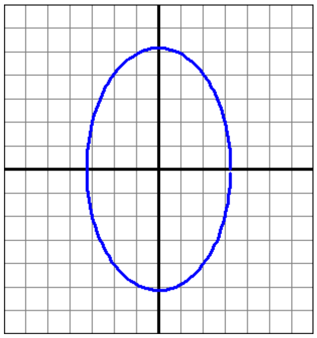
$$5x + 2y = 12 \quad \Rightarrow \quad y = f(x) = -\tfrac{5}{2}x + 6.$$

However, not all equations involving x and y can easily be rearranged algebraically into an explicit version, and others cannot be written explicitly at all. We'll call these *implicit* equations. (See Examples below).

It is sometimes possible to find a derivative $\frac{dy}{dx}$ from an implicit equation. The process is called *implicit differentiation*.

Example A: Given the equation $5x^2 + 2y^2 = 53$, a) Verify that the point $(x, y) = (-3, 2)$ satisfies the equation. b) Use implicit differentiation to find $\frac{dy}{dx}$. c) Find the equation of the tangent to the curve at $(x, y) = (-3, 2)$.

answers: $-\frac{5x}{2y}$; $\frac{15}{4}x + \frac{53}{4}$



Example B: Given the equation $\ln(x - y) = xy$, a) Use implicit differentiation to find $\frac{dy}{dx}$. b) Find the equation of the tangent to the curve at $(x, y) = (1, 0)$. *answers:* $\frac{xy - y^2 - 1}{-1 - x^2 + xy}$; $\frac{1}{2}x - \frac{1}{2}$

Example C: Given the equation $x^2 y^3 = 1$, a) Use implicit differentiation to find $\frac{d^2 y}{dx^2}$. b) Solve for the explicit equation and find $\frac{d^2 y}{dx^2}$. c) Show that the results from (a) and (b) are equal. *answer:* $\frac{10y}{9x^2}$

Example D: Given the equation $2x + 3y = e^{\sin(xy)}$, find $\frac{dy}{dx}$. *answer:* $\frac{ye^{\sin(xy)}\cos(xy) - 2}{3 - xe^{\sin(xy)}\cos(xy)}$

Example A revisited: Suppose that the equation $5x^2 + 2y^2 = 53$ represents the path of an oval racetrack. The position coordinates x and y would each be a function of time t. Use implicit differentiation to find $\frac{dy}{dt}$ in terms of x, y and $\frac{dx}{dt}$. [We're finding the y-component of velocity!]